Riemann Surfaces

Example Sheet 3

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1. Suppose $\Omega \subseteq \mathbb{C}$ is a discrete additive subgroup. [Here, *discrete* means that the subspace topology on Ω is discrete.] Show that one of the following holds:

   (i) $\Omega = \{0\}$, or

   (ii) $\Omega = \mathbb{Z}\omega$ for some $\omega \neq 0$, or

   (iii) $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1, \omega_2 \neq 0$ and $\omega_2/\omega_1 \notin \mathbb{R}$.

2. Let f be a simply periodic analytic function on $\mathbb{C}$ with periods $\mathbb{Z}$. Suppose furthermore that $f(x+iy)$ converges uniformly in x to (possibly infinite) limits as $y \to \pm\infty$. Show that $f(z) = \sum_{k=-n}^{n} a_k e^{2\pi i k z}$, i.e. $f(z)$ has a *finite* Fourier expansion.

3. Let f be a non-constant elliptic function with respect to a lattice $\Lambda \subset \mathbb{C}$, and let $P \subset \mathbb{C}$ be a fundamental parallelogram. Using the argument principle and, if necessary, slightly perturbing P, show that the number of zeros of f in P is the same as the number of poles, both counted with multiplicities.

   [This is also a consequence of the valency theorem, but the point of this question is that this more direct argument via contour integration also works.]

4. With the notation of the previous question, let the degree of f be n, let $a_1, \ldots, a_n$ denote the zeros of f in a fundamental parallelogram P, and let $b_1, \ldots, b_n$ denote the poles (both with possible repeats). By considering the integral

$$\frac{1}{2\pi i} \int_{\partial P} z \frac{f'(z)}{f(z)} dz$$

   and, if required, also slightly perturbing P, show that

$$\sum_{j=1}^{n} a_j \; - \; \sum_{j=1}^{n} b_j \; \in \Lambda \, .$$

5. Suppose a is a complex number with $|a| > 1$. Show that any analytic function f on $\mathbb{C}_*$ with $f(az) = f(z)$ for all $z \in \mathbb{C}_*$ must be constant, but that there is a non-constant meromorphic function f on $\mathbb{C}_*$ with $f(az) = f(z)$ for all $z \in \mathbb{C}_*$.

6. Let $\wp(z)$ denote the Weierstrass $\wp$-function with respect to a lattice $\Lambda \subset \mathbb{C}$. Show that $\wp$ satisfies the differential equation $\wp''(z) = 6\wp(z)^2 + A$, for some constant $A \in \mathbb{C}$. Show that there are at least three points and at most five points (modulo Λ) at which $\wp'$ is not locally injective.

7. With notation as in the previous question, and a a complex number with $2a \notin \Lambda$, show that the elliptic function

$$h(z) = (\wp(z-a) - \wp(z+a))(\wp(z) - \wp(a))^2 - \wp'(z)\wp'(a)$$

has no poles on $\mathbb{C} \smallsetminus \Lambda$. By considering the behaviour of h at $z = 0$, deduce that h is constant, and show that this constant is zero.

8. Find an explicit regular covering map of Riemann surfaces $\mathbb{D} \to \mathbb{D}_*$, where $\mathbb{D}$ denotes the open unit disc, as usual, and $\mathbb{D}_*$ denotes the punctured disc.

9. Show that $\mathbb{C} \smallsetminus \{P, Q\}$, where $P \neq Q$, is not conformally equivalent to $\mathbb{C}$ or $\mathbb{C}_*$, and deduce from the uniformization theorem that it is uniformized by the open unit disc $\mathbb{D}$. Show that the same is true for any domain in $\mathbb{C}$ whose complement has more than one point.

10. Let R be a compact Riemann surface of genus g and let $p_1, \dots, p_n$ be distinct points of R with $n \geq 1$. Show that $R \smallsetminus \{p_1, \dots, p_n\}$ is uniformized by the open unit disc $\mathbb{D}$ if and only if $2g - 2 + n > 0$, and by $\mathbb{C}$ if and only if $2g - 2 + n = 0$ or -1.

11. Let f, g be non-constant meromorphic functions on a compact Riemann surface R. Show that there is a non-zero polynomial $P(w_1, w_2)$ such that $P(f, g) = 0$.

    [Hint: Suppose f, g have valencies m, n respectively, and put $d = m + n$. Show that it is possible to choose complex numbers a_{ij}, not all zero, such that the function
    $$\sum_{j=0}^{d} \sum_{k=0}^{d} a_{jk} f(z)^j g(z)^k$$
    has at least $(d^2 + 2d)$ distinct zeros in R. Show that it cannot have more than d^2 poles, and deduce that it must be identically zero on R.]

12. Fix $d \geq 2$ and let $F'_d := \{(x, y) \in \mathbb{C}^2 : x^d + y^d = 1\}$. Show that the coordinate projections give F'_d the structure of a Riemann surface, and use topological gluing to find a compact Riemann surface F_d into which F'_d analytically embeds. Prove that the coordinate maps extend to meromorphic functions on F_d.